Energy spectrum evolution of a diffuse field in an elastic body caused by weak nonlinearity

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We study the evolution of diffuse elastodynamic spectral energy density under the influence of weak nonlinearity. It is shown that the rate of change of this quantity is given by a convolution of the linear energy at two frequencies. Quantitative estimates are given for sample aluminum and fused silica blocks of experimental interest.

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I. INTRODUCTION

Weak nonlinearity is known to cause a redistribution of elastodynamic spectral energy density. Energy present at one or two base frequencies will migrate, under the influence of nonlinearity, to higher harmonics and to sum and difference frequencies. If those sum and difference frequencies originally have little or no energy, their presence can be a sensitive measure of nonlinearity. The method has been shown to be capable of detecting flaws in otherwise linear-response specimens. It has been applied to crack detection [1] and to assessing the quality of adhesive bonds [2]. Much of this work has been qualitative, with emphases on how the nonlinear effects scale with field amplitude. The geometry is such, and the fields sufficiently uncontrolled, that attempts to quantify the nonlinearity in absolute terms have not been feasible.

In special circumstances, with high amplitude and long distance plane wave propagation, it is possible to monitor harmonic generation, and thereby assess nonlinearity, and do so quantitatively. It has been suggested that nonlinear Rayleigh wave propagation may be a useful NDE technique to measure surface properties [3]. Lamb waves were also suggested for this purpose [4].

As an alternative to plane wave configurations diffuse elastodynamic fields might be appropriate for NDE measurements under insufficiently controlled conditions. Diffuse fields span a broad range of applications from optics and microwaves to ultrasonics (see Ref. [5], and references therein). Gross properties of a specimen can be evaluated from integral field parameters [6]. The extension of diffuse field theory to include nonlinear effects is receiving increasing experimental [7] and theoretical [8] attention. Its application to the evaluation of material properties may be a future research focus.

The purpose of this paper is to describe effects of weak nonlinearity on the energy spectrum of a diffuse field, as one of the integral quantities available for experimental measurement. With a view toward eventual application in the context of NDE, but with an interest in the general problem, we take elastodynamics in a nonlinear solid as an underlying physical system for our diffuse field model.

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We start with the governing equations for the chosen field type in Sec. II, and develop them into a system of ODE's with quadratic and cubic nonlinear terms describing time evolution of the modal amplitudes. In Sec. III we discuss statistical assumptions for the normal frequencies and modes of the solid. In Sec. IV the nonlinear equations are treated by means of regular perturbation theory and averaged to obtain the spectral power transfered into a frequency band. Results independent of the underlying physical nature of the nonlinearity are considered. In Sec. V we specialize to the case of an isotropic homogeneous elastic body and an initial field generated by two transient narrow-band signals centered at different frequencies. Quantitative estimates for the nonlinear energy transfer into double and combination frequencies are provided and discussed. Conclusions are presented in Sec. VI.

II. GOVERNING EQUATIONS

Suppose we have a finite hyperelastic body occupying volume V_0 in its natural state referenced by coordinate $\mathbf{x}$, and having material density $\rho_0(\mathbf{x})$. The strain energy density of the body for a displacement field $\mathbf{u}$ with a corresponding Green's tensor $E_{ij}=1/2(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})$ is described to leading orders in strain by [9]

$$W=\frac{1}{2!}C_{ijkl}(\mathbf{x})E_{ij}E_{kl}+\frac{1}{3!}D_{ijklmn}(\mathbf{x})E_{ij}E_{kl}E_{mn}+\frac{1}{4!}F_{ijklmnpq}(\mathbf{x})E_{ij}E_{kl}E_{mn}E_{pq}+\cdots, \tag{1}$$

where $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{F}$ are linear, second, and third-order nonlinear elastic tensors, respectively.

Let the body be subjected to external forces and tractions that stop acting after a certain cutoff time $t_0=0$ after which the field in the body freely evolves without dissipation under zero-displacement boundary conditions. According to the Hamilton principle, the weak form of the governing equation for the field evolution can be written as

$$\int\int_{V_0}(-\sigma_{ji}\delta u_{i,j}-\rho_0\ddot{u}_i\delta u_i)d\mathbf{x}^3dt=0, \tag{2}$$

where σ is the first Piola-Kirchoff stress tensor. It is related to the strain energy density (1) via deformation gradient $F_{ij}=\delta_{ij}+u_{i,j}$ [9]:

$$\sigma_{ji}(\mathbf{x},\mathbf{u}) = \partial W/\partial F_{ij}. \tag{3}$$

To distinguish the parts of Eq. (2) responsible for linear and leading-order nonlinear behaviors of the system we expand the stress tensor σ in powers of $\|\mathbf{u}\|$ up to third order, and label the respective linear and nonlinear operators as $\hat{\mathbf{L}}$ and $\hat{\mathbf{N}}$. Equation (2) is then written as

$$\int\int_{V_0} [\delta u_i(\rho_0 \ddot{u}_i - \hat{L}_i\mathbf{u}) + \delta u_{i,j}\hat{N}_{ij}\mathbf{u}]d\mathbf{x}^3 dt = 0. \tag{4}$$

A complete set of eigenvalues ω_n^2 and their normalized eigenfunctions $\mathbf{u}^n$ is considered to be known for the linear operator as a solution of

$$\hat{L}_i\mathbf{u}^n \equiv \frac{\partial}{\partial x_j}C_{ijkl}u^n_{k,l} = -\rho_0\omega_n^2 u_i^n. \tag{5}$$

For clarity of notation we introduce composite Greek indices with implied summation upon them:

$$g_{\alpha\equiv\{\mathbf{x},i,j\}}h_{\alpha\equiv\{\mathbf{x},i,j\}} = \int_{V_0} g_{ij}(\mathbf{x})h_{ij}(\mathbf{x})d\mathbf{x}^3,$$

and denote the first partial derivatives of the field as separate functions $u_{\alpha\equiv\{\mathbf{x},i,j\}} = \partial u_i(\mathbf{x})/\partial x_j$.

The displacement field allows decomposition upon the eigenfunctions with time-dependent modal amplitudes d_k [10]

$$u_i(\mathbf{x},t) = \sum_k d_k(t)u_i^k(\mathbf{x}).$$

By employing this representation in Eq. (4), and using the eigenfunctions as a set of test functions $\delta\mathbf{u}$, we restate the governing equation as

$$\ddot{d}_k + \omega_k^2 d_k = -\sum_{m,l} N_{\alpha\beta\gamma}U^{klm}_{\alpha\beta\gamma}d_k d_l d_m - \sum_{m,l,n} N_{\alpha\beta\gamma\delta}U^{klmn}_{\alpha\beta\gamma\delta}d_k d_l d_m d_n, \tag{6}$$

where $U^{klm}_{\alpha\beta\gamma} = u^k_\alpha u^l_\beta u^m_\gamma$ and $U^{klmn}_{\alpha\beta\gamma\delta} = u^k_\alpha u^l_\beta u^m_\gamma u^n_\delta$. All the specifics of particular type of nonlinear behavior are now contained in matrices **N**,**U**.

Only symmetrical with respect to the last two indices parts of matrix $N_{\alpha\beta\gamma}$ and symmetrical with respect to the last three indices parts of matrix $N_{\alpha\beta\gamma\delta}$ survive modal summation in Eq. (6). Without loss of generality we put these matrices equal to their corresponding symmetrical parts, and write them in Greek index notation as

$$N_{\alpha\equiv\{\mathbf{x},i,j\}\beta\equiv\{\mathbf{x}',k,l\}\gamma\equiv\{\mathbf{x}'',m,n\}} = \frac{1}{2}N_{ijklmn}\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}'')$$

and

$$N_{\alpha\equiv\{\mathbf{x},i,j\}\beta\equiv\{\mathbf{x}',k,l\}\gamma\equiv\{\mathbf{x}'',m,n\}\delta\equiv\{\mathbf{x}''',p,q\}} = \frac{1}{6}N_{ijklmnpq}\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}'')\delta(\mathbf{x}-\mathbf{x}''').$$

Nonlinear terms of the stress (3) yield directional tensors in the above formulas

$$N_{ijklmn} = C_{ijln}\delta_{km} + C_{jnkl}\delta_{im} + C_{jlmn}\delta_{ik} + D_{ijklmn} \tag{7}$$

and

$$N_{ijklmnpq} = C_{jlnq}\delta_{ik}\delta_{mp} + C_{jnlq}\delta_{im}\delta_{kp} + C_{jqnl}\delta_{ip}\delta_{mk} + D_{jlmnpq}\delta_{ik} + D_{jnklpq}\delta_{im} + D_{jqmnkl}\delta_{ip} + D_{ijlnpq}\delta_{mk} + D_{ijlqmn}\delta_{pk} + D_{ijqnkl}\delta_{pm} + F_{ijklmnpq}.$$

Due to the major and minor symmetries of elastic tensors **C**, **D**, and **F** [9] both matrices **N** can be identified as fully symmetric:

$$N_{\alpha\beta\gamma} = N_{\beta\alpha\gamma} = N_{\gamma\beta\alpha},$$

$$N_{\alpha\beta\gamma\delta} = N_{\beta\alpha\gamma\delta} = N_{\gamma\beta\alpha\delta} = N_{\delta\beta\gamma\alpha}. \tag{8}$$

A quantity which we term linear energy stored in a single mode k at time t is

$$E_k = \frac{1}{2}(\dot{d}_k^2 + \omega_k^2 d_k^2). \tag{9}$$

In the absence of nonlinearity it is equal to the total energy of a mode, and is constant over time. It can be written in terms of the complex amplitudes ψ_k:

$$E_k^{(0)} = \frac{1}{2}|\psi_k|^2\omega_k^2. \tag{10}$$

The amplitudes ψ_k arise from the action of external forces and tractions prior to the cutoff time t_0. They describe evolution of the linear part of the field by means of the modal amplitudes $d_k^{(0)}$ that are found as solutions of the linearized version of equation (6), with matrices **N** put to zero [11]:

$$d_k^{(0)} \equiv a_k(t) = \mathrm{Im}\psi_k e^{-\imath\omega_k t}. \tag{11}$$

The energy flow due to nonlinearity $\dot{E}_k = \Pi_k$ is obtained from the governing equation (6), with modal power input being

$$\Pi_k = -N_{\alpha\beta\gamma}\sum_{m,l} U^{klm}_{\alpha\beta\gamma}\dot{d}_k d_l d_m - N_{\alpha\beta\gamma\delta}\sum_{m,l,n} U^{klmn}_{\alpha\beta\gamma\delta}\dot{d}_k d_l d_m d_n. \tag{12}$$

It must be noted that the energy quantity E_k (9) is not conserved, because it is not a true energy in the nonlinear case. Analysis of the strain energy density (1) reveals that additional terms $-N_{\alpha\beta\gamma}\Sigma_{m,l}U^{klm}_{\alpha\beta\gamma}d_k d_l d_m/3 - N_{\alpha\beta\gamma\delta}\Sigma_{m,l,n}U^{klmn}_{\alpha\beta\gamma\delta}d_k d_l d_m d_n/4$ must be added to E_k to produce a quantity that is conserved. However, the terms involve summation upon modal amplitudes other than that of

the mode at hand, and do not allow simple interpretation in terms of a single mode. Thus they are not used.

III. STATISTICAL MODEL

We restrict ourselves to a class of systems for which the field excited by external forces and tractions has a fully diffuse nature. In experiments such fields are practically realized, for example, in an elastic solid of a classically chaotic shape, and have statistical properties close to or indistinguishable from those of a field described by a random Hamiltonian [12,13]. The normal modes of the system are taken to be centered Gaussian vectors with a certain spatial correlation, as was first theoretically conjectured [14], and later numerically and experimentally verified [15]. We assume that the mean density of states of the normal frequencies is given by the function $D(\omega)$ in the form of Weyl-series [12], and frequency-frequency correlations can be neglected.

Since experimental identification of a particular mode is complicated by the "missing level" effect or modal overlap we choose to pursue calculation of average spectral density rather than individual modal amplitudes. In order not to distinguish between individual modes at the frequencies of interest, and thus deal with the average quantities, we limit observation time of the system t to be less than the corresponding break (Heisenberg) time $t_H=2\pi D$. On the other hand, time t is considered larger than transition (ballistic) time in the solid (mean time between two successive scattering events at the boundaries) $t_l=l/c$, so that diffuse regime of the field is established. l and c stand for characteristic diameter of and wavespeed in the solid. Putting together the two bounds yields $t_l \ll t \ll t_H$, a condition that can be experimentally realized.

In the framework of the adopted statistical model the modes $\mathbf{u}^k$ of the linear operator (5) are centered Gaussian random vectors with variance given by

$$\langle u_i^k(\mathbf{x}) u_j^n(\mathbf{x}')\rangle = \delta_{kn} K_{ij}(\omega_k, \mathbf{x}, \mathbf{x}'),$$

where $\langle\cdots\rangle$ represents ensemble average and $\mathbf{K}$ is a smooth frequency-dependent correlation matrix. Pairwise correlation of the first partial derivatives of the modes

$$\langle u_\alpha^k u_\beta^n\rangle = \delta_{kn} K'_{\alpha\beta}(\omega_k) \tag{13}$$

is readily obtained from

$$K'_{\alpha\equiv\{\mathbf{x},i,l\}\beta\equiv\{\mathbf{x}',j,m\}} = \partial^2 K_{ij}(\omega,\mathbf{x},\mathbf{x}')/\partial x_l\, \partial x'_m. \tag{14}$$

Orthonormality of the modes imposes a normalization condition upon the correlation matrix

$$\int_{V_0} \rho_0(\mathbf{x}) K_{ii}(\omega,\mathbf{x},\mathbf{x}) d\mathbf{x}^3 = 1. \tag{15}$$

From the mode statistics the complex amplitudes ψ_k describing initial field are found to be centered Gaussian random numbers with variance

$$\langle \psi_k \psi_n\rangle = 0, \frac{1}{2} D(\omega_k)\langle \psi_k^* \psi_n\rangle = \delta_{nk}\varepsilon(\omega_k).$$

The amplitudes ψ_k relate to the linear energy (10), and give the smooth function $\varepsilon(\omega_k)=D(\omega_k)\langle E_k^{(0)}/\omega_k^2\rangle$ an interpretation as a frequency-normalized spectral energy density. The linear amplitudes a_k (11) form a centered Gaussian ensemble as well. Their pairwise time correlation is calculated as

$$D(\omega_k)\langle a_k(t) a_n(\tau)\rangle = \delta_{kn}\varepsilon(\omega_k)\cos\,\omega_k(t-\tau). \tag{16}$$

Both the mean density of states and correlation matrix can be calculated in terms of the average Green's function $D(\omega)=2\omega\mathrm{Im}[\mathrm{Tr}\langle\mathbf{G}(\omega)\rangle]/\pi$, and $\mathbf{K}(\omega)=2\omega\mathrm{Im}\langle\mathbf{G}(\omega)\rangle/\pi D(\omega)$ [16]. For time scales under consideration, in particular $t_l \ll t_H$, the characteristic wavelength of the frequencies of interest is much smaller than the diameter of the solid $\lambda/l \ll 1$. The leading order nonlinear contribution thus comes from the bulk rather than near-boundary region of the solid. This allows us to neglect effects of the latter, and approximate the exact Green's function $\mathbf{G}$ in the solid by the Green's function in an unbounded medium $\mathbf{G}^\infty$. This approximation implies that $\mathbf{K}$ has an infinite correlation radius inherited from $\mathbf{G}^\infty$, and leads to formal integral divergence in the calculations of the following sections. To mend the problem we consider the scattering of the field inside the solid as a diffuse process with a free mean path on the order of the diameter l, which now provides a finite correlation radius for the model: $\langle\mathbf{G}\rangle=\langle\mathbf{G}^\infty\rangle e^{-|\mathbf{x}-\mathbf{x}'|/l}$ [17]. The ansatz is justified as the final results turn out not to depend on the specific choice of the value of l, as long as it stays much greater than the wavelength.

IV. ENERGY SPECTRUM EVOLUTION

We assume nonlinear effects to be small and seek solution of Eq. (6) by utilizing small perturbation theory, and expanding amplitudes d_k in orders of magnitude $d_k=a_k+b_k+(\cdots)$, where a_k are given by Eq. (11). Next order amplitudes b_k arise from the presence of nonlinearity, and are determined solely by the linear field

$$b_k(t) = -\int_{t_0}^{t} d\tau \frac{\sin\,\omega_k(t-\tau)}{\omega_k}\left[\sum_{m,l} N_{\alpha\beta\gamma} U^{klm}_{\alpha\beta\gamma} a_l(\tau) a_m(\tau) + \sum_{m,l,n} N_{\alpha\beta\gamma\delta} U^{klmn}_{\alpha\beta\gamma\delta} a_l(\tau) a_m(\tau) a_n(\tau)\right].$$

To obtain successive corrections to the average spectral density of the power flow we expand Eq. (12) in a series of magnitudes of a_k:

$$\Pi(\omega_k,t) \equiv D(\omega_k)\langle\Pi_k(t)\rangle = \sum_n \Pi^{(n)}(\omega_k,t). \tag{17}$$

As mentioned in Sec. II, since the power flow arises from the nonlinear mode coupling, the modal energy of the linear field (10) is conserved: $\Pi^{(0)}(\omega,t)=0$. For the assumed statistics of

the amplitudes a_k (16) the first-order correction to the average power density is zero as well:

$$\Pi^{(1)}(\omega_k,t) = D(\omega_k)\sum_{l,m} N_{\alpha\beta\gamma}\langle U^{klm}_{\alpha\beta\gamma}\rangle\langle \dot{a}_k(t)a_l(t)a_m(t)\rangle = 0.$$

The power flow expansion (17) starts with $\Pi^{(2)}$ as the leading term. We express the fourth moments of the amplitudes a_k as double products of their pairwise correlations (16), and obtain the power flow in terms of the energy density

$$\Pi^{(2)}(\omega_k,t) = D(\omega_k)\left\{\sum_{l,m} N_{\alpha\beta\gamma}N_{\nu\mu\eta}\langle U^{kll}_{\alpha\beta\gamma}U^{kmm}_{\nu\mu\eta}\rangle\varepsilon'(\omega_m)\varepsilon'(\omega_m)\frac{\sin\omega_k t}{\omega_k} + \frac{1}{2}\sum_{l,m,\pm} N_{\alpha\beta\gamma}N_{\nu\mu\eta}\langle U^{klm}_{\alpha\beta\gamma}U^{klm}_{\nu\mu\eta}\rangle\varepsilon'(\omega_l)\right.$$
$$\left.\times\left[\varepsilon'(\omega_m)\pm 2\varepsilon'(\omega_k)\frac{\omega_k}{\omega_m}\right]\left[\frac{\sin(\omega_k-\omega_l\pm\omega_m)t}{\omega_k-\omega_l\pm\omega_m}+\frac{\sin(\omega_k+\omega_l\pm\omega_m)t}{\omega_k+\omega_l\pm\omega_m}\right]\right\}, \tag{18}$$

where $\varepsilon'(\omega)=\varepsilon(\omega)/D(\omega)$ is the average normalized energy of a single mode. The modal sum is evaluated as a frequency integral with integrand weighed by the mean density of states

$$\sum_n f_n = \int_0^{+\infty} f(\omega_n)D(\omega_n)d\omega_n.$$

We note that matrix $N_{\alpha\beta\gamma\delta}$ responsible for the cubic nonlinearity does not enter Eq. (18), for its contribution is proportional to $\langle \dot{a}_k(t)a_l(t)a_m(t)a_n(t)\rangle=0$.

The centered Gaussian statistics of the modes allows averages in the form of $\langle U^{klm}_{\alpha\beta\gamma}U^{pqr}_{\nu\mu\eta}\rangle$ to be expressed as triple products of the pairwise correlations (13). Combining terms with the same modal indices we formally write (with no implied summation on Latin indices)

$$N_{\alpha\beta\gamma}N_{\nu\mu\eta}\langle U^{klm}_{\alpha\beta\gamma}U^{klm}_{\nu\mu\eta}\rangle = \mathbb{N}_0(\omega_k,\omega_m,\omega_l) + \mathbb{N}_2(\omega_k,\omega_l)\delta_{km} + \mathbb{N}_2(\omega_m,\omega_k)\delta_{lm} + \mathbb{N}_2(\omega_l,\omega_m)\delta_{lk} + \mathbb{N}_3(\omega_k)\delta_{kl}\delta_{km}. \tag{19}$$

The coupling functions $\mathbb{N}$ are defined later.

Expression (18) with the factors (19) inserted, though bulky and cumbersome to analyze, gives the leading term in the power flow due to a weak nonlinearity. In experimental practice, however, it is not uncommon to deal with the fields that have the spectral energy density ε varying smoothly on frequency scales $\Delta\omega$ greater than the averaging bandwidth of the limited observation time $\Delta\omega t\gg 1$. In this case, Eq. (18) simplifies further. All the terms directly proportional to rapidly oscillating sine factors yield negligible averages

$$\sin(\omega_n t)/\omega_n = D(\omega_n)O(1/\Delta\omega t)$$

and the modal sums involving such factors are evaluated as follows:

$$\sum_n f_n\sin[(\omega_n-\omega)t]/(\omega_n-\omega) = \pi D(\omega)f(\omega)[\theta(\omega) + O(1/\Delta\omega t)],$$

where θ is a unit step function. Another simplification arises if we pay attention only to the frequencies that carried no energy initially: $\varepsilon(\omega)=0$. For such frequencies a weak change in the energy density due to $\Pi^{(2)}$ is not masked by a strong initial linear field, and is convenient for experimental measurement.

With the simplifications mentioned the power flow expression reduces to

$$\Pi^{(2)}(\omega,t) = \frac{\pi}{2}\sum_{\pm}\int_0^{+\infty} D(\omega)\mathbb{N}_0(\omega,\omega',|\omega\pm\omega'|) \times\varepsilon(|\omega\pm\omega'|)\varepsilon(\omega')d\omega'. \tag{20}$$

The only remaining coupling function is given by contraction of the nonlinear and correlation matrices

$$\mathbb{N}_0(\omega_1,\omega_2,\omega_3) = N_{\alpha\beta\gamma}N_{\nu\mu\eta}K'_{\alpha\nu}(\omega_1)K'_{\beta\mu}(\omega_2)K'_{\gamma\eta}(\omega_3). \tag{21}$$

We note that

$$\mathbb{N}_0(\omega,\omega',|\omega+\omega'|) = \mathbb{N}_0(\omega,\omega'',|\omega-\omega''|)_{\omega''=\omega+\omega'}.$$

The power flow in the form of Eq. (20) allows a simple interpretation: the energy transfered into a given frequency ω comes from all pairs of frequencies ω' and ω'' that have the given frequency as a combination, i.e., equal to their sum or difference: $\omega=|\omega'\pm\omega''|$. The qualitative result is in agreement with and could have been expected from an elementary theory of nonlinear oscillations [18]. The symmetry of the nonlinear matrix (8) leads to the symmetry of $\mathbb{N}_0$ with respect to any interchange of its arguments. The coupling strength of any triad of frequencies is thus independent of the energy transfer direction. Nevertheless, the power itself exhibits an overall trend of the energy to be transfered up the frequency spectrum, as it is proportional to the density of states at the target frequency $D(\omega)$.

The expression (20) implies energy growth that is, if ε's are approximately constant in time, proportional to elapsed time t. In discrete spectrum systems such behavior is found when a triad of frequencies is locked in internal resonance, producing secular terms in the solution obtained by regular perturbation theory. This is the behavior observed in our case because of the finite time $t\ll t_H$: since individual modes are not resolved, any combination frequency produced by the

TABLE I. Mechanical properties.

	ρ_0 (kg/m^3)	c_l (m/s)	c_t (m/s)	λ (GPa)	μ (GPa)	A (GPa)	B (GPa)	C (GPa)
Aluminum[a]	2720	6100	3090	49.1	26.0	−320	−198	−190
Fused silica	2200[b]	5700	3750	9.60[b]	30.9[b]	−44[c]	93[c]	27[c]

[a]Alloy D54S, Smith *et al.* [21].
[b]Drumheller [22].
[c]Bechmann *et al.* [23].

source frequencies in the frequency range of interest is indistinguishable from at least one of the normal frequencies of the system, which hence lies in effective internal resonance with them.

V. ESTIMATES FOR ISOTROPIC HOMOGENEOUS ELASTIC SOLID

We return to elastodynamic displacement fields and specialize to the case of an elastic body composed of known isotropic homogeneous material, the case that holds premium experimental and theoretical interest. The linear part of the Green's function in the unbounded medium yields the correlation matrix **K** (see the Appendix):

$$K_{ij}(\Delta\mathbf{x},\omega)=\frac{1}{M}e^{-|\Delta\mathbf{x}|/l}\frac{1}{1/c_l^3+2/c_t^3}\times\left\{\frac{\delta_{ij}}{3}\left[\frac{1}{c_l^3}j_0(k_l|\Delta\mathbf{x}|)+\frac{2}{c_t^3}j_0(k_t|\Delta\mathbf{x}|)\right]-(\delta_{ij}/3-\Delta\hat{x}_i\Delta\hat{x}_j)\left[\frac{1}{c_l^3}j_2(k_l|\Delta\mathbf{x}|)-\frac{1}{c_t^3}j_2(k_t|\Delta\mathbf{x}|)\right]\right\},\tag{22}$$

where $\Delta\mathbf{x}=\mathbf{x}-\mathbf{x}'$ is a separation variable, $\Delta\hat{\mathbf{x}}=\Delta\mathbf{x}/|\Delta\mathbf{x}|$, M is total mass of the solid, c_l and c_t are the longitudinal and transverse wavespeeds, respectively, and j_n is the spherical Bessel function of order n. The mean density of states for clamped boundary conditions is calculated in Ref. [19]:

$$D(\omega)=\frac{V_0}{2\pi^2}\omega^2[1/c_l^3+2/c_t^3]-\frac{S_0}{8\pi c_l^2}\omega[2+(c_l/c_t)^2+3(c_l/c_t)^4]/[(c_l/c_t)^2+1]+O(l/c),$$

with S_0 being the surface area of the solid.

For the model of nonlinearity at hand appropriate for an isotropic solid, described by the five-constant theory [20], and correlation matrix **K** provided by Eq. (22), the coupling function $\mathbb{N}_0$ (21) assumes the form

$$N_0(\omega,\omega',|\omega-\omega'|)=\frac{\pi}{(\gamma_l^3+2\gamma_t^3)^3}\frac{c}{V_0M}\omega^3\times\left|1-\frac{\omega'}{\omega}\right|\frac{\omega'}{\omega}\tilde{\mathbb{N}}(\omega'/\omega).$$

$\tilde{\mathbb{N}}$ is a dimensionless function of the frequency ratio and material properties that characterize nonlinear coupling strength between the source and target frequencies. The function is independent of the linear dimensions of the body and possesses the symmetries according to Eq. (8): $\tilde{\mathbb{N}}(\omega'/\omega)=\tilde{\mathbb{N}}(\omega/\omega')=\tilde{\mathbb{N}}(1-\omega'/\omega)$. An analytical expression for $\tilde{\mathbb{N}}$ is available, but too bulky to be presented here. Details of the calculations are found in the Appendix.

As a sample distribution of the (linear) energy density we take two Gaussian peaks with half-width $\Delta\omega$ and total energies $E_{1,2}$ centered at frequencies ω_1 and ω_2:

$$\varepsilon(\omega)=\sum_{i=\{1,2\}}\frac{E_i}{\omega_i^2\sqrt{2\pi\Delta\omega^2}}e^{-(\omega-\omega_i)^2/2\Delta\omega^2}.\tag{23}$$

The width of the peaks is restricted by condition $\Delta\omega t\gg 1$, as imposed by applicability requirements for the simplified power input expression (20). It is additionally assumed that the peaks do not overlap: $|\omega_1-\omega_2|\gg\Delta\omega$. The suggested form of the spectrum closely models the combined spectra of two narrow-band signals that might be used in an experiment. By adjusting the carrying frequencies the features of nonlinear mode coupling can be investigated, and the form of $\tilde{\mathbb{N}}$ measured.

As a parameter suitable for characterization of the nonlinear energy transfer strength we consider transfer times defined as the formal time required for entire mean energy of the source peaks (23) to be transfered into combination frequencies, provided small perturbation theory and Eq. (20) holds: $t_{1,2}=E_{1,2}/\Pi^{(2)}_{1,2}$, and $t_{1\pm2}=\sqrt{E_1E_2}/\Pi^{(2)}_{1\pm2}$. Here $\Pi^{(2)}$ represents the total power input into frequency band supporting the resulting peak at the combination frequency. According to the definition $t_{1,2}$, t_{1+2}, and t_{1-2} characterize energy transfer into double, sum, and difference frequency, respectively. Calculations yield these times to be

$$t_{1\pm2}=\frac{1}{\pi}\frac{\omega_1^2\omega_2^2}{Mc^2D(|\omega_1\pm\omega_2|)\mathbb{N}_0(|\omega_1\pm\omega_2|,\omega_1,\omega_2)}\frac{1}{m_{1\pm2}^2},$$

$$t_{1,2}=\frac{2}{\pi}\frac{\omega_{1,2}^4}{Mc^2D(2\omega_{1,2})\mathbb{N}_0(2\omega_{1,2},\omega_{1,2},\omega_{1,2})}\frac{1}{m_{1,2}^2}.\tag{24}$$

As a measure of absolute strength of the linear field we choose squares of the Mach numbers $m_{1\pm2}^2=\sqrt{E_1E_2}/Mc^2$ and $m_{1,2}^2=E_{1,2}/Mc^2$. As seen from Eq. (24) stronger nonlinear effects (shorter times) require higher Mach numbers, i.e., stronger linear fields.

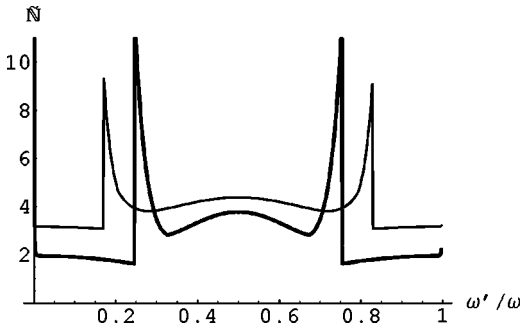

FIG. 1. Dimensionless coupling function in aluminum (thick line) and fused silica (thin line).

For numerical estimates we take an aluminum block and a fused silica block with the mechanical properties listed in Table I, and choose the transverse wavespeed for each material as the characteristic one: $c=c_t$. Due to the inversion symmetry we evaluate the coupling function $\tilde{\mathbb{N}}$ for frequency range $0<\omega'/\omega<1$ only, and plot it in Fig. 1. Immediately a significant feature of the plot comes into view: the sharp peaks and discontinuities of the coupling function at frequency ratios of 0.25 and 0.75 for aluminum, and 0.17 and 0.83 for fused silica.

The feature may be understood by considering nonlinear combinations of plane waves. We recall that from a plane wave perspective appropriate in the limit $\lambda/l \ll 1$, in addition to internal frequency resonance between the source frequencies ω' and $\omega''=|\omega-\omega'|$ and the target frequency ω, there is an additional requirement for wave vector resonance. We note that there are two plane wave types in the system, namely, longitudinal and transverse. If the types of the three waves (at ω, ω', and ω'') are identical, wave vector resonance $[\mathbf{k}(\omega'')=\mathbf{k}(\omega)+\mathbf{k}(\omega')$ with $|\mathbf{k}|=\omega/c]$ is always possible, and demands that the three wave vectors are parallel or antiparallel. If the wave type for one of the frequencies is different from the other two, then wave vector resonance is not always possible; it depends on the frequency ratio. If it is possible, there will be a nontrivial angle between the wave vectors. The transition between possibility and impossibility occurs at certain special frequency ratios $\omega/\omega'=(1\pm c_t/c_l)/2$. At these ratios wave number resonance occurs with parallel or antiparallel wave vectors. The special frequency ratio depends solely on the Poisson ratio of the material. For nondispersive single wave speed systems, when $c_l=c_t$, the ratios become 0 and 1. In this case the effect is not observed, as one of the source frequencies needs to be zero [24]. However, for certain dispersive single wave speed systems the peaks and discontinuities might still be found, if the dispersion equation of the system is such that the wave number resonance is possible. It is also worth noting that the discontinuity position at leading order is independent of the field strength. Thus field calibration is not required in experimental measurements in order to observe this characteristic feature, for only relative values of $\tilde{\mathbb{N}}$ are needed. This makes the method convenient to use in the light of its eventual application to NDE.

In order to see whether the presented theory is applicable for experimental verification and eventual use, we need to provide numerical estimates for transfer times (24). For this purpose we choose the solid to be equivalent in volume and surface area to a cube with a side of 7 cm. The typical values of the transition (ballistic) time in the system are of the order 10 μs. The carrying frequencies of the two narrow-band signals are taken to be $\omega_1/2\pi=500$ kHz and $\omega_2/2\pi=600$ kHz, so that the break (Heisenberg) time is of the same order as the absorption times of 100 ms common to experiment. The transfer times are calculated for the aluminum block and plotted in Fig. 2 as functions of the linear field strength, characterized by Mach number.

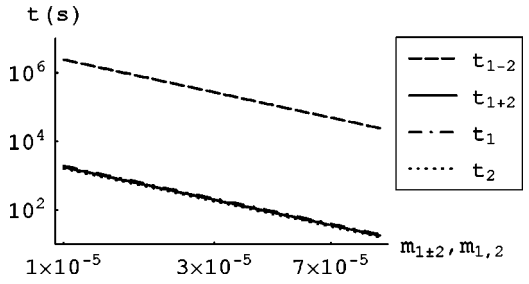

FIG. 2. Transfer times in aluminum block for fixed $\omega_1/2\pi=500$ kHz and $\omega_2/2\pi=600$ kHz.

It is feasible to generate linear diffuse fields with r.m.s. elastic strains corresponding to Mach numbers of order 10^{-5} that yield transfer times of order 10^3 s (see Fig. 2). Thus the observation times of the order of the absorption time yield energy densities at combination frequencies some 10^4 times weaker than the source signal energy, i.e., r.m.s. strains of 10^{-7}. These are easily detectable. We also note that the strain ratio (combination field to initial field) is much smaller than unity, so the use of regular perturbation theory is valid for the given time scale.

Finally, we calculate transfer times in aluminum and fused silica as a function of source frequency ratio for a fixed Mach number (see Fig. 3). The frequency dependence of the nonlinear coupling strength in terms of the transfer times reveals the frequency dependence of $\tilde{\mathbb{N}}$. The dips in the transfer times correspond to the peaks of the coupling function, and might be used to estimate Poisson's ratio of the material, or as a signature of nonlinearity. We notice that the energy transfer into near-zero frequencies is inefficient, as manifested by high difference frequency transfer times in the vicinity of $\omega_1/\omega_2=1$.

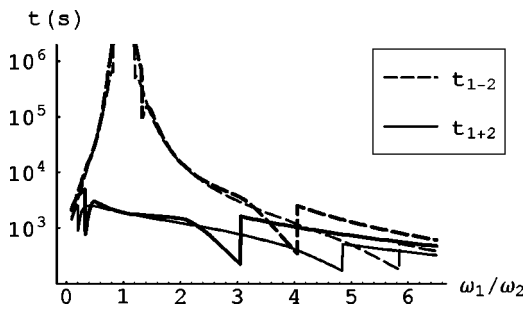

FIG. 3. Transfer times in aluminum (thick lines) and fused silica (thin lines) block for varying source frequency ratio, and fixed $\omega_2/2\pi=550$ kHz and $m_{1\pm2}=10^{-5}$.

VI. CONCLUSION

In the present work we have obtained a formula for the evolution of the average linear energy spectrum (20) of a weakly nonlinear system. Except for the definition of the modal linear energy E_k (9), and statistical properties of the normal frequencies ω_k and modes $\mathbf{u}^k$ (5), the result has no reference to the physical nature of the dynamic system governed by nonlinear equations (6). With proper generalization of the matrices $\mathbf{N},\mathbf{U}$ describing the physical nature of the nonlinearity it might be applicable beyond the scope of elasticity.

We observe that the energy redistribution occurs for triads of frequencies, with one being the sum or difference of the other two. The average power input into a narrow frequency band is found to be a cumulative effect coming from such interactions in the initial energy spectrum of the field, and to be proportional to the convolution of the energies stored at two frequencies. Relative weight of the interactions is given by the frequency-dependent coupling function $\mathbb{N}_0$ (21), and is calculated as a contraction of the nonlinear and correlation matrices $\mathbf{N}$ and $\mathbf{K}'$.

The case of chief experimental interest involving an isotropic homogeneous elastic body with nonlinearity given by the five-constant theory and an initial energy spectrum consisting of two narrow-band signals is discussed in detail. We find that characteristic times for the full energy transfer from the source into combination frequencies depend on the ratio of the source frequencies and exhibit characteristic dips and peaks and discontinuities at special frequency ratios corresponding to wavenumber resonance. The position of the resonance depends on Poisson's ratio. We also find that the transfer times are reciprocal to the strength of the initial signal given by the square of the corresponding Mach number.

The current theory is derived for zero-displacement boundary conditions of a clamped solid, and is impractical for experimental realization. However, for the frequencies of interest the main contribution to the mixing comes from the bulk of the solid. Since the near-boundary regions play a lesser role, the authors believe that the energy spectrum evolution for the traction-free elastic solid would exhibit behavior similar to that discussed here, and thus be accessible for experimental verification.

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APPENDIX: CORRELATION MATRIX AND COUPLING FUNCTION FOR ISOTROPIC HOMOGENEOUS MEDIUM

Elastic deformation of an isotropic homogeneous solid is described by the five-constant theory in terms of the Lamé constants λ and μ, and nonlinear coefficients A, B, and C [20]. According to the theory the linear and second-order nonlinear elastic tensors have the form

$$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + 2\mu\Phi^1_{ijkl},$$

$$D_{ijklmn} = 2A\Phi^2_{ijklmn} + 2C\delta_{ij}\delta_{kl}\delta_{mn} + 2B(\delta_{ij}\Phi^1_{klmn} + \delta_{kl}\Phi^1_{ijmn} + \delta_{mn}\Phi^1_{ijkl}). \tag{A1}$$

The elementary isotropic tensors Φ are given as follows:

$$\Phi^1_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$$\Phi^2_{ijklmn} = \frac{1}{8}(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{in}\delta_{jl}\delta_{km}).$$

By substituting tensors (A1) into the strain energy definition (1), particular form of directional tensors N_{ijklmn} (7) is derived.

The Green's function in the unbounded medium $\mathbf{G}^\infty$ is calculated from its spatial Fourier-transform [25]

$$G^\infty_{ij}(\mathbf{p},\omega) = \frac{\hat{p}_i\hat{p}_j}{c_l^2p^2-\omega^2} + \frac{\delta_{ij}-\hat{p}_i\hat{p}_j}{c_t^2p^2-\omega^2}, \quad \hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|.$$

Direct integration of the above expression and its subsequent normalization (15) yields particular expression for correlation matrix $\mathbf{K}$ (22). As two of its particular limit cases we note, first, the known autocorrelation function for scalar Helmholtz equation $j_0(k|\Delta\mathbf{x}|)$ [5] obtained by letting $c=c_l=c_t$. Second, the autocorrelation function for purely transverse field—such as, for example, electromagnetic field [26]—obtained by letting $c_l\to\infty$.

The correlation matrix of the first partial derivatives of the modes is derived from $\mathbf{K}$ by means of Eq. (14). It is expressed in terms of known directional tensors $\mathbf{H}$, characteristic wave number $k=\omega/c$, and wave speed ratios $\gamma_{l,t}=c/c_{l,t}$:

$$K'_{\alpha\equiv\{\mathbf{x},i,l\}\beta\equiv\{\mathbf{x}',j,m\}} = \frac{1}{M}e^{-|\Delta\mathbf{x}|/l}\frac{1}{\gamma_l^3+2\gamma_t^3}k^2 \times \sum_{\substack{a=\{0,2,4\}\\ \mathfrak{p}=\{l,t\}}} \gamma_p^5 H^{(\mathfrak{p},a)}_{ijmn} j_a(\gamma_p k|\Delta\mathbf{x}|). \tag{A2}$$

The sum over longitudinal and transverse wave types is denoted as $\mathfrak{p}=\{l,t\}$. Directional tensors $\mathbf{H}$ are defined by the following expressions:

$$H^{(l,0)}_{ijmn} = \frac{1}{3}\delta_{ij}\delta_{mn} - H^{(t,0)}_{ijmn} = \frac{1}{15}[\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}],$$

$$H^{(l,2)}_{ijmn} = \delta_{ij}Q^2_{mn} - H^{(t,2)}_{ijmn} = \frac{1}{7}Q^2_{ijmn},$$

$$H^{(l,4)}_{ijmn} = -H^{(t,4)}_{ijmn} = \frac{1}{7}Q^2_{ijmn} - Q^4_{ijmn}.$$

The quadruple, composite quadruple and 2^4-order directional moments $\mathbf{Q}$ moments yield zero values when integrated upon

all spatial directions or contracted upon any two pairs of indices:

$$Q_{ij}^2 = \frac{\delta_{ij}}{3} - \Delta\hat{x}_i \Delta\hat{x}_j,$$

$$Q_{ijmn}^2 = \delta_{ij}Q_{mn}^2 + \delta_{mn}Q_{ij}^2 + \delta_{im}Q_{jn}^2 + \delta_{jn}Q_{im}^2 + \delta_{in}Q_{jm}^2 + \delta_{jm}Q_{in}^2,$$

$$Q_{ijmn}^4 = \frac{1}{15}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) - \Delta\hat{x}_i\Delta\hat{x}_j\Delta\hat{x}_m\Delta\hat{x}_n.$$

The coupling function (21) is given by contraction of the directional tensors of the nonlinear matrix (7) with correlation matrices of the first derivatives (A2). It is used to find the dimensionless function

$$\tilde{\mathbb{N}} = \sum_{\substack{a,b,c=\{0,2,4\} \\ \mathfrak{p},\mathfrak{q},\mathfrak{r}=\{l,t\}}} (\gamma_\mathfrak{p}\gamma_\mathfrak{q}\gamma_\mathfrak{r})^4 \mathbb{H}_{abc}^{\mathfrak{pqr}} I_{abc}^{\mathfrak{pqr}}(\omega'/\omega),$$

where constants $\mathbb{H}$ are

$$\mathbb{H}_{abc}^{\mathfrak{pqr}} = N_{ijklmn}N_{pqrstu}H_{ipjq}^{(\mathfrak{p},a)}H_{krls}^{(\mathfrak{q},b)}H_{mtnu}^{(\mathfrak{r},c)},$$

and the integral I is given by

$$I_{abc}^{\mathfrak{pqr}}(\Omega) = \frac{\gamma_\mathfrak{p}\gamma_\mathfrak{q}}{\gamma_\mathfrak{r}^2}\Omega|1-\Omega|\int_0^{+\infty} z^2 e^{-z/3\gamma_\mathfrak{r}k} \times j_a\left(\frac{\gamma_\mathfrak{p}}{\gamma_\mathfrak{r}}\Omega z\right) j_b\left(\frac{\gamma_q}{\gamma_r}|1-\Omega|z\right) j_c(z)dz. \quad \text{(A3)}$$

For a finite correlation radius of the coupling matrix l, the integral additionally depends on the target frequency ω. The dependence turns out to be only significant in the small vicinity of $\Omega=\omega'/\omega=\{0,1\}$, elsewhere the contribution being small: $O(1/kl)$. We note that for the mentioned ratios Ω one of the source frequencies ω' or $|\omega-\omega'|$ must be close to zero, and the phenomenon is of small practical importance from experimental point of view. For these near-zero frequencies the wavelength becomes comparable to or greater than the diameter of the solid. Thus the substitution of the exact Green's function **G** by the Green's function in the unbounded medium $\mathbf{G}^\infty$ is no longer valid, and conditions for the time scales made in Sec. III are not met.

With these limitations in mind we set $\lambda/l=0$ for practical calculations, and obtain analytical expression for Eq. (A3). As the result of this procedure a singularity of the function $\tilde{\mathbb{N}}$ at $\omega'/\omega=\{0,1\}$ is acquired (not shown in Fig. 1). Discontinuities of the function at the wave number resonances, and discontinuities of its slope at $\omega/\omega'=2/(1+c_l/c_t)$ and $1-2/(1+c_l/c_t)$, are found as well. We expect in practice to observe sharp transitions over finite ranges in ω/ω' of order $1/kl$, at a characteristic frequency ratio ω/ω' possibly shifted by an amount $O(1/kl)$.

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